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A Topological Study of Contextuality and Modality in Quantum Mechanics

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Abstract Kochen–Specker theorem rules out the non-contextual assignment of values to physical magnitudes. Here we enrich the usual orthomodular structure of quantum mechanical propositions with modal operators. This enlargement allows to refer consistently to actual and possible properties of the system. By means of a topological argument, more precisely in terms of the existence of sections of sheaves, we give an extended version of Kochen–Specker theorem over this new structure. This allows us to prove that contextuality remains a central feature even in the enriched propositional system.

Keywords Contextuality · Sheaves · Modal · Quantum logic

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1 Introduction

Modal interpretations of quantum mechanics [4–6, 10] face the problem of finding an objective reading of the accepted mathematical formalism of the theory, a reading "in terms of properties possessed by physical systems, independently of consciousness and measurements (in the sense of human interventions)" [6]. These interpretations intend to consistently include the possible properties of the system in the discourse looking for a new link between the state of the system and the probabilistic character of its properties, thus sustaining that the interpretation of the quantum state must contain a modal aspect. The name *modal interpretation* was for the first time used by B. van Fraassen [9] following *modal logic*, precisely the logic that deals with possibility and necessity. Within this frame, a physical property of a system means "a definite value of a physical quantity belonging to this system; i.e., a feature of physical reality" [6]. As usual, definite values of physical magnitudes correspond to yes/no propositions represented by orthogonal projection operators acting on vectors belonging to the Hilbert space of the (pure) states of the system [15].

Formal studies of modal interpretations of quantum logic exist which are similar to the modal interpretation of the intuitionistic logic [2, 11]. A description of such approaches may be found in [3]. At first sight, it may be thought that the enrichment of the set of (actual) propositions with modal ones could allow one to circumvent the contextual character of quantum mechanics. We have faced the study of this issue and given a Kochen–Specker type theorem for the enriched lattice [8]. In this paper, we give a topological version of that theorem, i.e. we study contextuality in terms of sheaves.

2 Basic Notions

We recall from [12, 18] and [19] some notions of sheaves and lattice theory that will play an important role in what follows. First, let $(A, \leq)$ be a poset and $X \subseteq A$. X is a *decreasing set* iff for all $x \in X$, if $a \leq x$ then $a \in X$. For each $a \in A$ we define the *principal decreasing set* associated to a as $(a] = \{x \in A : x \leq a\}$. The set of all decreasing sets in A is denoted by A^+, and it is well known that $(A^+, \subseteq)$ is a complete lattice, thus $\langle A, A^+\rangle$ is a topological space. We observe that if $G \in A^+$ and $a \in G$ then $(a] \subseteq G$. Therefore $B = \{(a] : a \in A\}$ is a base of the topology A^+ which we will refer to as the *canonical base*. Let I be a topological space. A *sheaf* over I is a pair (A, p) where A is a topological space and $p : A \to I$ is a local homeomorphism. This means that each $a \in A$ has an open set G_a in A that is mapped homeomorphically by p onto $p(G_a) = \{p(x) : x \in G_a\}$, and the latter is open in I. It is clear that p is a continuous and open map. *Local sections* of the sheaf p are continuous maps $\nu : U \to I$ defined over open proper subsets U of I such that $p\nu = 1_U$. In particular we use the term *global section* only when $U = I$.

In a Boolean algebra A, congruences are identifiable to certain subsets called *filters*. $F \subseteq A$ is a filter iff it satisfies: if $a \in F$ and $a \leq x$ then $x \in F$ and if $a, b \in F$ then $a \wedge b \in F$. Two comments are in order. On the one hand, if F is a filter, then $\Theta_F = \{(x, y) \in A^2 : (x \to y) \wedge (y \to x) \in F\}$ is a congruence. On the other hand, if Θ is a congruence, then $F_\Theta = \{x \in A : (x, 1) \in \Theta\}$ is a filter. It is not very hard to see that $\Theta_{F_\Theta} = \Theta$ and $F_{\Theta_F} = F$. Thus we establish the filter-congruence identification. F is a proper filter iff $F \neq A$ or, equivalently $0 \notin F$. If $X \subseteq A$, the filter F_X generated by X is the minimum filter containing X. A proper filter F is *maximal* iff the quotient algebra A/F is isomorphic to **2**, being **2** the two-elements Boolean algebra. It is well known that each proper filter can be extended to a maximal one.

We denote by $\mathcal{OML}$ the variety of orthomodular lattices. Let $L = \langle L, \vee, \wedge, \neg, 0, 1\rangle$ be an orthomodular lattice. Given a, b, c in L, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, (a,b,c)D^* hold for all permutations of a, b, c. An element z of a lattice L is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(L)$ the set of all central elements of L and it is called the *center* of L. $Z(L)$ is a Boolean sublattice of L [19, Theorem 4.15].

3 Sheaf-Theoretic View of Contextuality

Let $\mathcal{H}$ be the Hilbert space associated to the physical system and $L(\mathcal{H})$ be the set of closed subspaces on $\mathcal{H}$. If we consider the set of these subspaces ordered by inclusion, then $L(\mathcal{H})$ is a complete orthomodular lattice [19]. It is well known that each self-adjoint operator $\mathbf{A}$ that represents a physical magnitude A may be associated with a Boolean sublattice W_A of $L(\mathcal{H})$. More precisely, if A has discrete spectrum, then W_A is the Boolean algebra of projectors $\mathbf{P}_i$ of the spectral decomposition $\mathbf{A} = \sum_i a_i \mathbf{P}_i$ of the discrete operator $\mathbf{A}$. We will refer to W_A as the spectral algebra of the operator $\mathbf{A}$. Any value attributing proposition about the system is represented by an element of $L(\mathcal{H})$ which is the algebra of quantum logic introduced by G. Birkhoff and J. von Neumann [1].

Assigning values to a physical quantity A is equivalent to establishing a Boolean homomorphism $v : W_A \to \mathbf{2}$ [13]. Thus, it is natural to consider the following definition which provides us with a *compatibility condition*:

Definition 3.1 Let $(W_i)_{i\in I}$ be the family of Boolean sublattices of $L(\mathcal{H})$. A global valuation over $L(\mathcal{H})$ is a family of Boolean homomorphisms $(v_i : W_i \to \mathbf{2})_{i\in I}$ such that $v_i \mid W_i \cap W_j = v_j \mid W_i \cap W_j$ for each $i, j \in I$.

The Kochen–Specker theorem (KS) precludes the possibility of assigning definite properties to the physical system in a non-contextual fashion [17]. An algebraic version of KS theorem is given by [7, Theorem 3.2]:

Theorem 3.2 *If $\mathcal{H}$ be a Hilbert space such that* $\dim(\mathcal{H}) > 2$, *then a global valuation over $L(\mathcal{H})$ is not possible.*

It is also possible to give a topological version of this theorem in the frame of local sections of sheaves. In fact, let L be an orthomodular lattice. We consider the family $\mathcal{W}_L$ of all Boolean subalgebras of L ordered by inclusion and the topological space $\langle \mathcal{W}_L, \mathcal{W}_L^+\rangle$. On the set

$$E_L = \{(W, f) : W \in \mathcal{W}_L,\ f : W \to \mathbf{2},\ f \text{ is a Boolean homomorphism}\}$$

we define a partial ordering given by $(W_1, f_1) \le (W_2, f_2)$ iff $W_1 \subseteq W_2$ and $f_1 = f_2 \mid W_1$. Thus we can consider the topological space $\langle E_L, E_L^+\rangle$ whose canonical base is given by the principal decreasing sets $((W, f)]$ defined as $((W, f)] = \{(G, f \mid G) : G \subseteq W\}$. For simplicity $((W, f)]$ is noted as $(W, f]$.

We note that the map $p_L : E_L \to \mathcal{W}_L$ such that $(W, f) \mapsto W$ is a sheaf over $\mathcal{W}_L$. This sheaf is called the *spectral sheaf* associated to the orthomodular lattice L.

Let $\nu : U \to E_L$ be a local section of p_L. By [7, Proposition 4.2], for each $W \in U$ we have that $\nu(W) = (W, f)$ for some Boolean homomorphism $f : W \to \mathbf{2}$ and if $W_0 \subseteq W$, then $\nu(W_0) = (W_0, f \mid W_0)$. From a physical perspective, we may say that the spectral sheaf takes into account the whole set of possible ways of assigning truth values to the propositions associated with the projectors of the spectral decomposition $\mathbf{A} = \sum_i a_i \mathbf{P}_i$. The continuity of a local section of p guarantees that the truth value of a proposition is maintained when considering the inclusion of subalgebras. In this way, the compatibility condition 3.1 of the Boolean valuation with respect to the intersection of pairs of Boolean sublattices of $L(\mathcal{H})$ is maintained. Thus, continuous local sections of p_L are identifiable to compatible contextual valuations.

We use $\nu(a) = 1$ to note that there exists $W \in U$ such that $a \in W$, $\nu(W) = (W, f]$ and $f(a) = 1$. On the other hand, if $f : W \to 2$ is a Boolean homomorphism, $\nu : (W] \to E_L$ is such that for each $W_i \in (W]$, $\nu(W_i) = (W_i, f/W_i)$ is a local section of p_L. We call this a *principal* local section.

A global section $\tau : \mathcal{W}_L \to E_L$ of p_L is interpreted as follows: the map assigns to every $W \in \mathcal{W}_L$ a fixed Boolean valuation $\tau_w : W \to \mathbf{2}$ obviously satisfying the compatibility condition. Thus, KS theorem in terms of the spectral sheaf reads [7, Theorem 4.3]:

Theorem 3.3 *If $\mathcal{H}$ is a Hilbert space such that* $\dim(\mathcal{H}) > 2$ *then the spectral sheaf $p_{L(\mathcal{H})}$ has no global sections.*

4 An Algebraic Study of Modality

With these tools, we are now able to build up a framework to include modal propositions in the same structure as actual ones. To do so we enrich the orthomodular structure with a modal operator taking into account the following considerations: (1) Propositions about the properties of the physical system are interpreted in the orthomodular lattice of closed subspaces of the Hilbert space of the (pure) states of the system. (2) Given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it together with a set of propositions that are compatible with it and, at the same time, predicate probabilities about the other ones. In other words, one may predicate truth or falsity of all possibilities at the same time, i.e. possibilities allow an interpretation in a Boolean algebra. In rigorous terms, for each proposition P, if we refer with $\Diamond P$ to the possibility of P, then $\Diamond P$ will be a central element of the orthomodular structure. (3) If P is a proposition about the system and P occurs, then it is trivially possible that P occurs. This is expressed as $P \leq \Diamond P$. (4) Assuming an actual property and a complete set of properties that are compatible with it determines a context in which the classical discourse holds. Classical consequences that are compatible with it, for example probability assignments to the actuality of other propositions, shear the classical frame. These consequences are the same ones as those which would be obtained by considering the original actual property as a possible one. This is interpreted in the following way: if P is a property of the system, $\Diamond P$ is the smallest central element greater than P. From consideration (1) it follows that the original orthomodular structure is maintained. The other considerations are satisfied if we consider a modal operator $\Diamond$ over an orthomodular lattice L defined as $\Diamond a = \mathrm{Min}\{z \in Z(L) : a \leq z\}$ with $Z(L)$ the center of L under the assumption that this minimum exists for every $a \in L$.

Let A be an orthomodular lattice. We say that A is *Boolean saturated* if and only if for all $a \in A$ the set $\{z \in Z(A) : z \leq a\}$ has a maximum [7]. In this case, the maximum is denoted by $\Box(a)$. This is a particular case of Janowitz's quantifier in orthomodular lattices [14]. In view of [19, Lemma 29.16], complete orthomodular lattices with an operator $e(a) = \bigvee\{z \in Z(L) : z \leq a\}$, are examples of Boolean saturated orthomodular lattices. They conform a variety of algebras $\langle A, \wedge, \vee, \neg, \Box, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$, noted as $\mathcal{OML}^\Box$ [8]. $\mathcal{OML}^\Box$ are axiomatized as follows:

S1	Axioms of $\mathcal{OML}$	S5	$\Box(x \wedge y) = \Box(x) \wedge \Box(y)$
S2	$\Box x \leq x$	S6	$y = (y \wedge \Box x) \vee (y \wedge \neg\Box x)$
S3	$\Box 1 = 1$	S7	$\Box(x \vee \Box y) = \Box x \vee \Box y$
S4	$\Box\Box x = \Box x$	S8	$\Box(\neg x \vee (y \wedge x)) \leq \neg\Box x \vee \Box y$

For the axioms of $\mathcal{OML}$, we refer to [16].

On each algebra of $\mathcal{OML}^\Box$ we can define the *possibility operator* as the unary operation $\Diamond$ given by $\Diamond x = \neg\Box\neg x$. It satisfies $a \leq \Diamond a$ and $\Diamond a = \mathrm{Min}\{z \in Z(A) : a \leq z\}$. If L is an orthomodular lattice then there exists an orthomodular monomorphism $f : L \to A$ such that $A \in \mathcal{OML}^\Box$ [8, Theorem 10]. We refer to A as a *modal extension* of L. In this case we can see the lattice L as a subset of A. If $L^\Box \in \mathcal{OML}^\Box$ is a modal extension of L, we define the *possibility space* of L in $L^\Box$ as $\Diamond L = \langle\{\Diamond p : p \in L\}\rangle_{L^\Box}$. If W is a Boolean sublattice of L then $\langle W \cup \Diamond L\rangle_{L^\Box}$ is a Boolean sublattice of $L^\Box$; in particular $\Diamond L$ is a Boolean sublattice of $Z(L^\Box)$ [8, Theorem 14]. The possibility space represents the modal content added to the discourse about properties of the system.

5 Sheaves and Modality

Let us consider $L^\Box$ a modal extension of L. Then, the spectral sheaf p_L is a subsheaf of $p_{L^\Box}$. In this case we refer to $p_{L^\Box}$ as a *modal extension* of p_L. It is clear that local sections of p_L can be seen as local sections of $p_{L^\Box}$. We define the set

$$Sec(\Diamond L) = \{\nu : (\Diamond L] \to E_{L^\Box} : \nu \text{ is principal section of } p_{L^\Box}\}.$$

Since $\Diamond L$ is a Boolean algebra, it is a subdirect product of **2**. Thus, it always exists a Boolean homomorphism $f : \Diamond L \to 2$, resulting $Sec(\Diamond L) \neq \emptyset$. From a physical point of view, $Sec(\Diamond L)$ represents all physical properties as possible properties. The fact that $Sec(\Diamond L) \neq \emptyset$ shows that, *in the frame of possibility*, one may talk simultaneously about all physical properties.

In the orthomodular lattice of the properties of the system, it is always possible to choose a context in which any possible property pertaining to this context can be considered as an actual one. We formalize this fact in the following definition and then we prove that this is always possible in our modal structure:

Definition 5.1 Let L be an orthomodular lattice, W a Boolean sublattice of L, $q \in W$ and $L^\Box$ be a modal extension of L. If $\nu \in Sec(\Diamond L)$ such that $\nu(\Diamond q) = 1$ then an actualization of q compatible with ν is an extension $\nu' : U \to E_{L^\Box}$ such that $(\langle W \cup \Diamond L\rangle_{L^\Box}] \in U$.

Theorem 5.2 *Let L be an orthomodular lattice, W a Boolean sublattice of L, $q \in W$ and $L^\Box$ be a modal extension of L. If $\nu \in Sec(\Diamond L)$ such that $\nu(\Diamond q) = 1$ then there exists an actualization of q compatible with ν.*

Proof Suppose that $\nu(\Diamond L) = (\Diamond L, f)$. Let F be the filter associated with the Boolean homomorphism f. We consider the $\langle W \cup \Diamond L\rangle_{L^\Box}$-filter F_q generated by $F \cup \{q\}$. F_q is a proper filter. In fact: if F_q is not proper, then there exists $a \in F$ such that $a \wedge q \leq 0$. Thus $q \leq \neg a$ being $\neg a$ a central element. But $\Diamond q$ is the smallest Boolean element greater than q. Then $\Diamond q \leq \neg a$ or equivalently $\Diamond q \wedge a = 0$. And this is a contradiction since $\Diamond q,\ a \in F$. Thus, we may extend F_q to be a maximal filter F_M in $\langle W \cup \Diamond L\rangle_{L^\Box}$. If we consider the natural projection $f_{F_M} : \langle W \cup \Diamond L\rangle_{L^\Box} \to \langle W \cup \Diamond L\rangle_{L^\Box}/F_M \approx \mathbf{2}$, the local section $\nu' : (\langle W \cup \Diamond L\rangle_{L^\Box}] \to E_{L^\Box}$ is an actualization of q compatible with ν. □

The next theorem allows a representation of the Born rule in terms of continuous local sections of sheaves. This rule quantifies possibilities from a chosen spectral algebra.

Theorem 5.3 *Let L be an orthomodular lattice, W a Boolean sublattice of L, and $\nu : (W] \to E_L$ a principal local section. If we consider a modal extension $L^\Box$ of L then there exists an extension $\nu' : U \to E_{L^\Box}$ such that $\langle W \cup \Diamond L\rangle_{L^\Box} \in U$.*

Proof Suppose that $\nu(W) = (W, f)$. Let $i : W \to \langle W \cup \Diamond L\rangle_{L^\Box}$ be the Boolean canonical embedding. We see that there exists a Boolean homomorphism $f' : \langle W \cup \Diamond L\rangle_{L^\Box} \to \mathbf{2}$ such that $f = f'i = f' \mid W$ since $\mathbf{2}$ is injective in the variety of Boolean algebras [20]. Thus $\nu' : (\langle W \cup \Diamond L\rangle_{L^\Box}] \to E_{L^\Box}$ is the extension required. □

We note that this reading of the Born rule is a kind of the converse of the possibility of actualizing properties given by Theorem 5.2.

Definition 5.4 Let L an orthomodular lattice, $L^\Box$ be a modal extension and $\nu \in Sec(\Diamond L)$. An actualization compatible with ν is a global section $\tau : \mathcal{W}_L \to E_L$ of p_L such that $\tau(W \cap \Diamond L) = \nu(W \cap \Diamond L)$.

Theorem 5.5 *Let L be an orthomodular lattice. Then p_L is a global section τ iff for each modal extension $L^\Box$ there exists $\nu \in Sec(\Diamond L)$ such that τ is a compatible actualization of ν.*

Proof Suppose that p_L admits a global section $\tau : \mathcal{W}_L \to E_L$ and let $\tau(W) = (W, f_W)$. We consider the family $(A_W = W \cap \Box L)_{W \in \mathcal{W}_L}$. Let $f_0 : \bigcup_W A_W \to \mathbf{2}$ such that $f_0(x) = f_W(x)$ if $x \in W$. f_0 is well defined since τ is a global section. If we consider $\langle \bigcup_W A_W \rangle_{L^\Box}$ the subalgebra of $L^\Box$ generated by the join of the family $(A_W)_W$, it may be proved that it is a Boolean subalgebra of $\Diamond L$. We can extend f_0 to a Boolean homomorphism $f_0' : \langle \bigcup_W A_W \rangle_{L^\Box} \to \mathbf{2}$. Since $\mathbf{2}$ is injective in the variety of Boolean algebras [20], then there exists a Boolean homomorphism $f : \Diamond L \to \mathbf{2}$ which extends it to f_0'. If we consider $\nu \in Sec(\Diamond L)$ such that $\nu(G) = (G, f \mid G)$, it result that τ is a compatible actualization of ν. The converse is immediate. □

To conclude we may say that the addition of modalities to the discourse about the properties of a quantum system enlarges its expressive power. At first sight it may be thought that this could help to circumvent contextuality, allowing to refer to physical properties belonging to the system in an objective way that resembles the classical picture. In view of the last theorem, since any global section of the spectral sheaf is a compatible actualization of a local one belonging to $Sec(\Diamond L)$, a global actualization that would correspond to a family of compatible valuations is prohibited. Thus, the theorem states that the contextual character of quantum mechanics is maintained even when the discourse is enriched with modalities.

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